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BIVARIATE MODELLING OF STOCHASTIC FEATURES: A CONVEX MIXTURE COPULA APPROACH

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ABSTRACT

The Clayton and Joe copulas serve as mathematical constructs utilised within copula theory to explicate the interrelationships among stochastic variables. Renowned for their efficacy in capturing extreme tail occurrences, these copulas are recognised members of the Archimedean copula family. Their utility spans diverse domains: genetics, neuroscience, finance, insurance, hydrology, environmental research, telecommunications, and reliability engineering. To augment their flexibility, we present in this discourse an elevated mixture of the Clayton and Joe copulas, denoted as the CJM copula. This convex mixture offers a direct linear interpolation between the Clayton and Joe copulas, modulated by a blending parameter. Such integration facilitates a more nuanced depiction of the copula density function. We scrutinise its attributes, including determining the blending parameter, Kendall's tau, and quadrant-dependent coefficients, while introducing three innovative copulas: the x-flipping CJM, y-flipping CJM, and survival CJM copulas. The collective discoveries significantly contribute to advancing the theoretical underpinnings governing copula-based modelling methodologies and their ensuing applications.

Keywords: Clayton Copula, Joe Copula, Convex Mixture, Dependence Parameter, Mixing Parameter

INTRODUCTION

The concept of copulas was initially introduced by Sklar in 1959 (Genest *et al.*, 2023; Sklar, 1959). Since then, it has evolved into a pivotal tool in statistical modelling, enabling the characterisation of dependent relationships among random variables (Klement *et al.*, 2005). Sklar's theorem posits that any multivariate distribution can be deconstructed into its marginal distributions and a copula function. This deconstruction empowers both practitioners and scholars to dissect specific dependency patterns autonomously, thereby fostering a more sophisticated comprehension of extensive and complex datasets (Alanazi, 2021; Chesneau, 2023; Yanan Fan *et al.*, 2020). Copulas have found diverse applications across various domains, including genetics (Trégouët *et al.*, 1999), neuroscience (Lee *et al*, 2019), finance (Boateng *et al.*, 2022), insurance (McNeil *et al.*, 2015), hydrology (De Luca *et al*, 2023; Han *et al.*, 2023a), environmental research (Pengfei *et al*, 2024), telecommunications (Duda *et al.*, 2024), and reliability engineering (LIU *et al.*, 2023) among others. The versatility of copulas facilitates the representation of relationships exhibiting diverse degrees of association, ranging from asymmetrical to non-linear (Chesneau, 2023).

There are two well-established copulas families(Mai *et al*, 2014). These include members of the Archimedean family (such as Joe, Ali-Mikhail-Haq, Clayton, Gumbel, Frank, and Independent) and the elliptical family (such as Gaussian and Student's-t). Sklar's theorem is applied to generate elliptical copulas from component-wise transformations of elliptically distributed random vectors. Interestingly, the ease of simulating from elliptical copulas is similar to that from elliptical distributions. Nonetheless, elliptical copulas' restriction to radial symmetry and lack of closed-form expressions are significant disadvantages.

In numerous financial and insurance contexts, an expectation arises for an elevated correlation between significant losses, such as those observed during stock market downturns, and substantial gains. However, the utilisation of elliptical copulas for modelling encounters challenges due to these inherent asymmetries, complicating the realistic representation of such nuanced relationships. Conversely, Archimedean copulas offer a broader spectrum of dependent structures, characterised by distinctive attributes facilitating the portrayal of tail dependencies and dependency structures. Moreover, standard Archimedean copulas exhibit closed-form expressions, rendering them particularly captivating subjects for scholarly inquiry (Embrechts *et al.*, 2003). Joe (1997), Nelson (2006), and Durante *et al.*(2016) provide insights into the properties and applications of various copulas. Consequently, what follows next are respectfully the definition of a copula, Archimedean copulas, convex finite linear mixture of copulas, probability density of a copula, and conditional distribution of a copula.

DEFINITIONS

Copula

A d–dimensional copula is a function $C: [0\,,1]^d \rightarrow [0\,,1]$ with the following properties:

P1: Grounded $C(u_1, \ldots, u_{i-1}, 0, u_{i+1}, \ldots, u_d) = 0$ for all $i = 1,\ldots,d$.

P2: Uniform univariate margins $C(1, \ldots, 1, u_i, 1, \ldots, 1) = u_i$ for all i=1,…,d.

P3: (d-increasing) for each rectangle $[a, b] := \prod_{i=1}^{d}[a_i, b_i] \subset [0,1]^d$ with $a_i \leq b_i \quad for\ all\ i = 1, \ldots, d,$

$$V_c([a,b]) := \sum_{j_1=1}^{2} \dots \sum_{j_d=1}^{2} (-1)^{j_1+\dots+j_d} C\left(u_{1_{j_1}}, \dots, u_{d_{j_d}}\right) \geq 0$$

where $u_{i1} = a_i$ and $u_{i2} = b_i$ for all i = 1,…,d.

We denote by C^d the set of all d-copulas.

(Flores *et al.*, 2017)

Archimedean copulas

The Archimedean copulas are specified as follows:

$$C(u_1, \dots, u_d) = \varphi^{-1}\{\varphi(u_1) + \dots + \varphi(u_d)\},$$

where $(u_1, \dots, u_d) \in [0,1]^d$ is a random variable. $\varphi(\cdot)$ and $\varphi^{-1}(\cdot)$ represent a generator function and its inverse respectively which satisfy the following conditions:

(i) φ is a continuous, strictly decreasing and convex function mapping [0,1]onto [0,∞],

(ii) $\varphi(0) = \infty$, and

(iii) $\varphi(1) = 0$

Archimedean copulas have dependence parameters, which describe the association between the random variables they model. The rest of the study will focus more on the Clayton and Joe mixture copula.

Convex finite linear mixture of copulas

Let $(u_1, \dots, u_d) \in [0,1]^d$ be a random variable and let C_i be d-dimensional copulas with *i =1,...d*. Then a convex finite linear mixture, C C_i is given by

$$C(u_1, \dots, u_d) = \sum_{i=1}^{d} \lambda_i C_i(u_1, \dots, u_d) \quad \textbf{eqn 1}$$

where $\lambda_i \geq 0$ for every $i \in [1,d]$ and $\sum_{i=1}^{d} \lambda_i = 1$ (Salvadori *et al.*, 2007; Song *et al.*, 2014). The convex finite linear mixture so defined is also a copula (Flores *et al.*, 2017; Nelson, 2006).

$$c(u_1, \dots, u_d) = \frac{\partial^d C(u_1, \dots, u_d)}{\partial u_1 \dots \partial u_d} \quad \textbf{eqn 2}$$

(Wiboonpongse *et al.*, 2015)

Probability density function of a copula

Let $C(u_1,\dots,u_d)$ be an absolutely continuous copula. Then its probability density function is defined as

Conditional distribution function of a copula

Let $C(u_1,\dots,u_d)$ be an absolutely continuous d-dimensional copula. Then the conditional distribution function of u_1 given $U_2 = u_2, \dots, U_d = u_d$ is defined as

$$C(u_1 | U_2 = u_2, \dots, U_d = u_d) = \frac{\partial^{d-1} C(u_1, \dots, u_d)}{\partial u_2, \dots, u_d} \times \frac{1}{c(u_1, \dots, u_d)} \quad \textbf{eqn 3}$$

(Sohrabian, 2021)

The above stated definitions shall be applied in the materials and methods section.

The Clayton and Joe copulas are discussed next to provide the impetus for the convex combination of the Clayton and Joe copula.

The Clayton Copula

The Clayton copula is a member of the Archimedean copula family. It is characterised by a Cumulative Distribution Function (CDF) given by:

$$C_C(u_1, \dots, u_d \ ; \delta) = \left[u_1^{-\delta} + \cdots + u_d^{-\delta} + 1 - d\right]^{-\frac{1}{\delta}}, \ \delta \ \in [-1, \infty\ ) \setminus 0] \quad \textbf{eqn 4}$$

where $(u_1,\dots,u_d) \in [0,1]^d$ is a d-dimensional random variable and write the delta symbol here is the dependence parameter. It has generator and inverse functions given by $\psi_\delta(t) = \frac{1}{\delta}\ (t^{-\delta} - 1)$ and $\psi_\delta^{-1}(t) = (1 + \ \delta t)^{-\frac{1}{\delta}}$ respectively.

The dependence parameter of the Clayton copula may be estimated from an expression between Kendall's tau and the dependence parameter given by = $\frac{\delta}{\delta+2}$.

The dependence parameter, δ, controls the strength of the lower tail dependence. It is well-suited for modelling lower tail dependence, making it particularly useful in capturing extreme events in the lower tails of bivariate distributions. It is skewed and, therefore, often used to model asymmetric dependencies. It is commonly applied in finance, risk management, and reliability analysis (Durante *et al*, 2016; Genest *et al*, 1993; Joe, 1997; Nelson, 2006).

The Joe Copula

The Joe copula is a member of the Archimedean copula family, providing a flexible framework for modelling various types of dependence structures. It is defined by a CDF given by

$$C_J(u_1, \dots, u_d \ ; \gamma) = n - 1 - \ \left\{\sum_{i=1}^{d}(1 - u_i)^\gamma \ - \ \prod_{i=1}^{d}(1 - u_i)^\gamma\right\}^{\frac{1}{\gamma}}, \gamma \ \in [1, \infty\ ) \quad \textbf{eqn 5}$$

where $(u_1,\dots,u_d) \in [0,1]^d$ as defined previously and γ is the dependence parameter.

The Joe copula has generator and inverse functions given by $\psi_\gamma(t) = -ln(1 - (1 - t)^\gamma)$ and $\psi_\gamma^{-1}(t) = 1 - (1 - \exp(-t))^{\frac{1}{\gamma}}$ respectively. The dependence parameter of the Joe copula may be estimated from an expression between Kendall's tau and the dependence parameter given by = $\frac{\gamma+3}{2(\gamma+2)^2}$.

The dependence parameter, γ, influences the tail dependence and overall shape of the copula. It is symmetric if γ=1 allowing for modelling symmetric dependence. However, it exhibits asymmetric dependence for other values of γ. It is versatile, accommodating a wide range of dependence patterns, including non-linear and skewed dependencies. It is commonly employed in risk management, reliability analysis, and finance in scenarios where capturing upper tail behaviour and extreme dependencies is essential (Joe, 1993, 1997; Nelson, 2006; Triantafillou, 2024).

The foregoing discussion of the Clayton and Joe copulas about their capabilities and inabilities provides the leverage for the enhancement of either of the two copulas. Boateng *et al.* (2022) also suggested that the tail dependence of existing Archimedean copulas needs to be enhanced.

A few of the literature on mixture copula mentioned the Clayton and Joe mixture copula (Caillault *et al.*, 2007; Chang, 2017; Patton, 2006; Yamaka *et al.*, 2023). However, none of the documents cited provided explicit formulae for the copula density, conditional distribution function, estimate for the mixing parameter as well as rotated copulas for the mixture of the Clayton and Joe copulas. This study therefore seeks to introduce the Clayton and Joe mixture copula to enhance their modelling capabilities by providing explicit formulae for the mixture copula, the density, conditional distribution and rotated copulas.

Now that we have laid the groundwork, the article will proceed to describe the Clayton and Joe mixture copula, outlining its primary attributes and derivative forms. We will then move on to discuss the results from our simulation studies before offering a conclusion and potential topics for further research.

MATERIALS AND METHODS

This part of the study is focused on providing the Clayton and Joe copulas with superior tail dependence capabilities through a convex mixture particularly sequel to (Boateng *et al.*, 2018, 2022). What follows next is a combination of the Clayton and Joe copulas using the definition of a convex mixture copula as provided in Equation (1) as a premise.

The Convex mixture of Clayton and Joe Copulas

Proposition 1: Define $(u_1,\ldots,u_d) \in [0,1]^d$ to be a d-dimensional random variable. Suppose $C_C(u_1,\ldots,u_d;\delta)$ and $C_J(u_1,\ldots,u_d;\gamma)$ are Clayton and Joe copulas with dependence parameters $\delta \in [-1,\infty)\backslash 0]$ and $\gamma \in [1,\infty)$ respectively. Then for any $\eta \in [0,1]$, the linear combination C_{CJ} as defined below is a copula.

$$C_{CJ}(u_1,\ldots,u_d;\delta,\gamma,\eta) = \eta C_C(u_1,\ldots,u_d;\delta) + (1-\eta)\, C_J(u_1,\ldots,u_d;\gamma), \quad \textbf{eqn 6}$$

where η is the mixing parameter.

Proof:

The proof is obvious from Equation (1) by taking $k = 2$, $\lambda_1 = \eta$, $\lambda_2 = 1 - \eta$, $C_1(u_1,\ldots,u_d) = C_C(u_1,\ldots,u_d;\delta)$ and $C_2(u_1,\ldots,u_d) = C_J(u_1,\ldots,u_d;\gamma)$

Now considering Proposition 1 in two dimensions, we have the following.

Let $(u,v) \in [0,1]^2$. Suppose δ and γ are as defined earlier, the dependence parameters for the Clayton and Joe copulas respectively and η the mixing parameter.

The 2-dimensional version of Equation (6) is given by

$$C_{CJ}(u,v;\,\delta,\gamma,\eta) = \eta(u^{-\delta}+v^{-\delta}-1)^{-\frac{1}{\delta}} + (1-\eta)\{1-[(1-u)^{\gamma}+(1-v)^{\gamma}-(1-u)^{\gamma}(1-v)^{\gamma}]^{\frac{1}{\gamma}}\} \quad \textbf{eqn 7}$$

We shall call Equation (7) the Clayton-Joe Mixture (CJM) copula.

The following propositions respectfully provide the probability density function, the conditional distribution function, Kendall's tau representation and an estimate of the mixing parameter of the CJM copula.

Probability density function

Proposition 2: Suppose δ and γ are the dependence parameters for the Clayton and Joe copulas respectively with random variables $(u,v) \in [0,1]^2$. Then for any mixing parameter $\eta \in [0,1]$, the probability density function of the CJM copula as stated in Equation (2) is given by

$$c_{CJ}(u,v;\ \delta,\gamma,\eta) = \eta\left\{(1+\delta)\left[u^{-\delta}+v^{-\delta}-1\right]^{-\left(\frac{1}{\delta}+2\right)}u^{-(\delta+1)}\ v^{-(\delta+1)}\right\}$$
$$+(1-\eta)\left[\!\left[[\gamma\,(1-u)^{\gamma-1}(1-v)^{\gamma-1}]\{(1-u)^{\gamma}+(1-v)^{\gamma}-(1-u)^{\gamma}(1-v)^{\gamma}\}^{\frac{1}{\gamma}-1}\right.\right.$$
$$+\left[\!\left[(1-\gamma)\{(1-u)^{\gamma}+(1-v)^{\gamma}-(1-u)^{\gamma}(1-v)^{\gamma}\}^{\frac{1}{\gamma}-2}\right.\right.$$
$$\left.\left. * \ \{(1-v)^{\gamma-1}[(1-u)^{\gamma}-1]\}\right]\!\right] * \{\,(1-u)^{\gamma-1}[1-\ (1-v)^{\gamma}]\}\ \left.\right]\!\right]$$

eqn 8

Conditional probability distribution

Proposition 3: Suppose δ and γ are the dependence parameters for the Clayton and Joe copulas respectively with random variables (u,v) ∈ $[0,1]^2$.Then for any mixing parameter η ∈ [0,1], the corresponding conditional probability distribution with respect to of the CJM copula stated in

Equation (3) is given by

$$C_{CJ}(v/u\,;\eta,\delta,\gamma\,) = \eta\left[u^{-\delta}+v^{-\delta}-1\right]^{-\frac{1}{\delta}-1}u^{-\delta-1}+(1-\eta)\ (-1)\left\{[(1-u)^{\gamma}+(1-v)^{\gamma}-\right.$$
$$\left.(1-u)^{\gamma}(1-v)^{\gamma}]^{\frac{1}{\gamma}-1}\right\}\{(1-u)^{\gamma-1}[(1-v)^{\gamma}-1]\}$$

(Han *et al.*, 2023b)

Kendall's tau for the CJM copula

Proposition 4: Suppose δ and γ are the dependence parameters for the Clayton and Joe copulas respectively.

Let the Kendall's tau for the bivariate Clayton and Joe copulas be given by $\frac{\delta}{\delta+2}$ and $\frac{\gamma+3}{2(\gamma+2)^2}$ respectively (Boateng *et al.*, 2018; Gari & Gelcho, 2022).

Then for any η ∈ [0,1], the Kendall's tau for the CJM copula is given by

$$\tau_{CJ} = \frac{2\eta\theta\,(\gamma+2)^2+\ (\delta+2)(\gamma+3)(1-\eta)}{2(\delta+2)\,(\gamma+2)^2}$$

Proof:

Let τ_C,τ_J and τ_{CJ} be the bivariate versions of the Kendall's tau for the CJM copulas respectively. Let δ and γ represent the dependence parameters of the Clayton and Joe copulas respectively. Let η be the mixing parameter of the CJM copula.

Then, given that $\tau_C = \frac{\delta}{\delta+2}$ and $\tau_J = \frac{\gamma+3}{2(\gamma+2)^2}$.

Kendall's tau for the CJM copula, C_{CJ}, is given by

$$\tau_{CJ} = \eta\frac{\delta}{\delta+2}+(1-\eta)\left(\frac{\gamma+3}{2(\gamma+2)^2}\right)$$
$$\tau_{CJ} = \frac{2\eta\delta\,(\gamma+2)^2+(\delta+2)(\gamma+3)(1-\eta)}{2(\delta+2)\,(\gamma+2)^2}$$

eqn 9

Estimation of the mixing parameter

Proposition 5: Suppose δ and γ are the dependence parameters for the Clayton and Joe copulas respectively. Let the Kendall's tau for the CJM copulas be τ_{CJ}. Then the estimate of the mixing parameter η is given by

$$\hat{\eta} = \frac{(\delta + 2)\left\{2\tau_{CJ}\,(\gamma + 2)^2 - (\gamma + 3)\right\}}{\left\{2\delta\,(\gamma + 2)^2 - (\delta + 2)(\gamma + 3)\right\}}$$

Proof:

From equation (9), we have

$$2\tau_{CJ}(\delta + 2)\,(\gamma + 2)^2 = 2\eta\delta\,(\gamma + 2)^2 + (\delta + 2)(\gamma + 3)(1 - \eta)$$

It follows therefore that

$$\hat{\eta} = \frac{(\delta+2)\left\{2\tau_{CJ}\,(\gamma+2)^2 - (\gamma+3)\right\}}{\left\{2\delta\,(\gamma+2)^2 - (\delta+2)(\gamma+3)\right\}} \qquad \textbf{eqn 10}$$

The following proposition provides the quadrant dependence coefficients for the CJM copula.

Quadrant dependence coefficients

Proposition 6: Let the CJM copula be given by

$$C_{CJ}(u, v;\ \delta, \gamma, \eta) = \eta\left[u^{-\delta} + v^{-\delta} - 1\right]^{-\frac{1}{\delta}} + (1 - \eta)\left\{1 - [(1 - u)^{\gamma} + (1 - v)^{\gamma} - (1 - u)^{\gamma}(1 - v)^{\gamma}]^{\frac{1}{\gamma}}\right\}$$

where $\eta \in [0,1]$, $(u,v) \in [0,1]^2$ and δ, and γ are the dependence parameters for the Clayton and Joe copulas respectively.

Then,

(i) The lower left tail dependent coefficient $\lambda_1 = 2^{-\frac{1}{\delta}}\eta$

(ii) The lower right tail dependent coefficient $\lambda_2 = 0$ and

(iii) The upper left tail dependent coefficient and $\lambda_3 = 0$

(iv) The upper right tail dependent coefficient $\lambda_4 = (2 - 2^{\frac{1}{\gamma}})(1 - \eta)$

Proof: Lower left tail dependence coefficient

Let λ_1 be the lower left tail dependence coefficient. Then, $\lambda_1 = \lim_{u \to 0^+} \frac{C_{CJ}(u,u)}{u}$ and so

$$\lambda_1 = \lim_{u \to 0^+} \frac{\eta[u^{-\delta} + u^{-\delta} - 1]^{-\frac{1}{\delta}} + (1 - \eta)\,\{1 - [2(1-u)^{\gamma} - (1-u)^{2\gamma}]^{\frac{1}{\gamma}}\}}{u}$$

Hence, $\lambda_1 = 2^{-\frac{1}{\delta}}\eta$

Proof: Lower right tail dependence coefficient

Let λ_2 be the lower right tail dependence coefficient. Then,

$$\lambda_2 = \lim_{u \to 0^+} \frac{u - C_{CJ}(1 - u, u)}{u}$$

$$\lambda_2 = \lim_{u \to 0^+} \frac{u - \eta\left[(1-u)^{-\delta} + u^{-\delta} - 1\right]^{-\frac{1}{\delta}} + (1-\eta)\left\{1 - [u^{\gamma} + (1-u)^{\gamma} - u^{\gamma}(1-u)^{\gamma}]^{\frac{1}{\gamma}}\right\}}{u}$$

$$\lambda_2 = 1 - [\eta * 1 + (1 - \eta) * 1] = 0$$

Proof: Upper left tail dependence coefficient

Let λ_3 be the upper left tail dependence coefficient.

Then,

$$\lambda_3 = \lim_{u \to 0^+} \frac{u - C_{CJ}(u, 1 - u)}{u}$$

Since C_{CJ} (1-u,u) is symmetric in u (Nelson, 2006), $\lambda_3 = \lambda_2 = 0$

Proof: Upper right tail dependence coefficient

Let λ_4 be the upper right tail dependence coefficient Then,

$$\lambda_4 = \lim_{u \to 1^-} \left[\frac{1 - 2u + C_{CJ}(u,u)}{1-u} \right]$$

Let u = 1 - w, then w = 1-u and when u = 0, w = 1 and when u = 1, w = 0.

So,

$$\lambda_4 = \lim_{u \to 1^-} \left[\frac{1 - 2u + C_{CJ}(u,u)}{1-u} \right] = \lim_{w \to 0^+} \left[\frac{1 - 2(1-w) + C_{CJ}(1-w, 1-w)}{w} \right]$$

$$\lambda_4 = (1-\eta)(2 - 2^{\frac{1}{\gamma}})$$

Derivative copulas

New copulas can be formed from the CJM copula using the procedures described in Nelson (2006) and Wiboonpongse *et al.* (2015) as illustrated below.

Given the CJM copula in equation [7], we introduce the following three derivative copulas:

(i) the x-flipping (90 degrees anti-clockwise rotated) CJM copula by

$$C_{CJ}(u,v) = v - C_{CJ}(1-u, v)$$

$$= v - \left[\!\left[\eta\left[(1-u)^{-\delta} + v^{-\delta} - 1\right]^{-\frac{1}{\delta}} + (1-\eta)\left\{1 - [u^{\gamma} + (1-v)^{\gamma} - u^{\gamma}(1-v)^{\gamma}]^{\frac{1}{\gamma}}\right\} \right]\!\right]$$

(ii) the y-flipping (270 degrees anti-clockwise rotated) CJM copula by

$$C_{CJ}(u,v) = u - C_{CJ}(u, 1-v)$$

$$= u - \left[\!\left[\eta\left[u^{-\delta} + (1-v)^{-\delta} - 1\right]^{-\frac{1}{\delta}} + (1-\eta)\left\{1 - [(1-u)^{\gamma} + v^{\gamma} - v^{\gamma}(1-u)^{\gamma}]^{\frac{1}{\gamma}}\right\} \right]\!\right]$$

(iii) the survival (180 degrees anti-clockwise rotated) CJM copula by

$$C_{CJ}(u,v) = u + v - 1 + C_{CJ}(1-u, 1-v)$$

$$= [\![u + v - 1]\!]$$

$$+ \left[\!\left[\eta\left[(1-u)^{-\delta} + (1-v)^{-\delta} - 1\right]^{-\frac{1}{\delta}} + (1-\eta)\left\{1 - [u^{\gamma} + v^{\gamma} - v^{\gamma}u^{\gamma}]^{\frac{1}{\gamma}}\right\} \right]\!\right]$$

(Embrechts *et al.*, 2001)

RESULTS AND DISCUSSION

In order to assess the performance of the CJM copula, simulation studies were undertaken using two marginal distributions. The first simulation was done with discrete-continuous random variables (i.e., Poisson (λ=5) and exponential (θ=0.05) margins); the second simulation was done with two continuous random variables (i.e, gamma (α=4;β=0.02) and exponential (θ=0.05) margins); and the third simulation was done with two discrete random variables (i.e., Poisson (λ=20) and negative binomial (μ=100;θ=4.5) margins). The simulation encompassed observations of varying sample sizes, specifically 100, 1,000, 10,000, and 100,000, respectively. The algorithm for the study follows next.

Algorithm for Modelling the dependence of two marginal distributions

1. Set seed (1357)
2. Select 100 observations each from Poisson and exponential distributions
3. Combine the observations for one of the bivariate copulas in R
4. Determine the AIC and BIC
5. Do same from step 1-4 using 1,000, 10,000, and 100,000 observations.
6. Do same from step 1-5 using the following distributions in step 2

   (i) Gamma and exponential distribution

   (ii) Poisson and negative binomial distribution
7. Do same from step 1-6 using each of the other 39 copulas
8. Do same from step 1-7 using the CJM copula
9. Arrange the results for each of the 40 copulas and that of the CJM in order of increasing AIC/BIC.

Tables 1, 2, and 3 present the results of the model comparison of the CJM copula against the best-selected bivariate copula from 40 bivariate copulas with Poisson and exponential, gamma and exponential, and Poisson and negative binomial margins, respectively.

Table 1: Table 1: Copulas with Poisson (λ=5) and Exponential (θ=0.05) margins

N	CJM & Best Bivariate Copula Selected	AIC	BIC
	Rotated Joe copula (270 degrees)	-3.1935	-0.5883
100	CJM	-29.16574	-23.9554
	Rotated Gumbel copula (90 degrees)	-2.2151	2.6927
1,000	CJM	-1415.565	-1405.75
	Independence copula	0.0000	0.0000
10,000	CJM	-14.43758	-0.016897
	Rotated Joe copula (270 degrees)	-0.6674	8.8455
100,000	CJM	-33.58137	-14.55552

From Table 1, it was found that for the Poisson and exponential simulation, at n = 100, 1,000, 10,000, and n = 100,000, rotated Joe (270 degrees), rotated Gumbel (90 degrees), independence, and rotated Joe (270 degrees), respectively, were the best copulas among the 40 bivariate copulas considered. However, when their AICs and BICs were compared with those of the CJM copula, the AIC and BIC of the CJM copula were smaller, hence superior, than the rotated Joe (270 degrees), rotated Gumbel (90 degrees), independence, and rotated Joe (270 degrees) copulas respectively. Similar results were obtained in the other simulations, where the CJM copula emerged superior to the best copulas among the 40 bivariate copulas considered. Tables 2 and 3 below provide the details.

Table 2: Copulas with Gamma (α = 4 ; β = 0.02) and Exponential (θ = 0.05) margins

n	CJM & Best Bivariate Copula Selected	AIC	BIC
	Rotated Joe copula (180 degrees)	-0.8312	1.7739
100	CJM	-24.6314	-19.42106
	Rotated Tawn type 1 copula (270 degrees)	-8.1588	1.6567
1,000	CJM	-9.812856	0.0026546
	Independence copula	0.0000	0.0000
10,000	CJM	-24.6314	-19.42106
	Independence copula	0.0000	0.0000
100,000	CJM	-14.41883	0.001847

Table 3: Copulas with Poisson (λ = 20) & Negative Binomial (μ = 100;θ = 4.5) margins

n	CJM & Best Bivariate Copula Selected	AIC	BIC
	Rotated Joe copula (90 degrees)	-1.9806	0.6246
100	CJM	-28.84852	-23.63818
	Rotated Tawn type 1 copula (90 degrees)	-2.9599	6.8556
1,000	CJM	-28.90462	-19.08911
	Independence copula	0.0000	0.0000
10,000	CJM	-14.42052	0.0001617
	Rotated Tawn type 1 copula (270 degrees)	-0.7123	18.3136
100,000	CJM	-19.02581	-4.605128

CONCLUSIONS

This article described a bivariate modelling of stochastic features through a convex mixture of Clayton and Joe copulas that retains all of the properties of a copula. It may serve as a Clayton or Joe copula depending on the value of the mixing parameter, increasing its versatility. This three-parameter copula may represent any characteristic that Clayton or Joe copulas can model. As a result, it outperforms both Clayton and Joe copulas.

The work investigated the probability density function, conditional distribution function, and various copulas generated from the Clayton and Joe Mixture copula. The study also included quadrant dependency coefficients, such as upper left tail, lower left tail, upper right tail, and lower right tail dependence coefficients. The study concluded with simulated examples investigating acceptable stochastic features -marginal distributions- that might operate with the CJM copula and its performance. Table 1 summarises the simulation results, demonstrating the superiority of the Clayton-Joe mixture (CJM) copula for modelling dependence in Poisson and exponential distributions compared to other bivariate copulas. For example, while the rotated Joe (270 degrees) copula performed best at n = 100, the rotated Gumbel (90 degrees) at n = 1,000, the independence copula at n = 10,000 and the rotated Joe (270 degrees) copula at n= 100,000, the CJM copula consistently exhibited lower AIC and BIC values across all sample sizes, thus indicating a better fit. This consistent CJM dominance was observed across all simulations, including those with exponential and gamma marginals (Table 2) and Poisson and negative binomial marginals (Table 3). These findings contribute to improved copula-based modelling.

Several promising avenues for future research are outlined below:

Topics for further research

Several new copulas can be derived through modification of Archimedean copulas and other copula families that may model some existing stochastic features better than what exists currently or perform just as the current copulas.

Another contribution will be to extend the current bivariate convex mixture to higher dimensions such as the trivariate variant indicated next.

$$C_{CJ}(u,v,w;\,\delta,\gamma,\eta) = \eta\left[u^{-\delta}+v^{-\delta}+w^{-\delta}-2\right]^{-\frac{1}{\delta}}$$

$$+(1-\eta)\left\{1-\left[(1-u)^{\gamma}+(1-v)^{\gamma}+(1-w)^{\gamma}-(1-u)^{\gamma}(1-v)^{\gamma}(1-w)^{\gamma}\right]^{\frac{1}{\gamma}}\right\}$$

Where $(u,v,w) \in [0,1]^3$, $\delta \in [-1,\infty)\backslash 0]$, $\gamma \in [1,\infty)$, $\eta \in [0,1]$.

Yet another contribution will be to apply this model to real-life data. This shall be our next direction on dependence modelling.

Declaration of conflict of interest

The authors declare that they have no known competing financial interest or personal relationship that could have appeared to influence the work reported in this paper.

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