AN IMPROVED MDOF MODEL SIMULATING SOME SYSTEM WITH DISTRIBUTED MASS

ENGINEERING

N.N. Osadebe, PhD
Department of Civil Engineering
University of Nigeria, Nsukka, Nigeria

ABSTRACT

This paper proposed and examined a MDOF model whose mass properties are determined on the hypothesis that at an arbitrary time in the course of the system's motion, the kinetic energy as well as the nodal displacements of the actual system (system with distributed mass) are the same as those of the simulating MDOF model. The solution of this formulation for any MDOF system with n degrees of freedom leads to n distinct mass vectors, each satisfying this condition of dynamic equivalence for a particular vibrational mode. Further more, for the conditions of dynamic equivalence to be satisfied the lumped masses of the simulating MDOF model must not be constant but are dependent upon the excitation frequencies. All the natural frequencies evaluated on the basis of this present formulation are almost identical with those of the actual system proving the high accuracy of the proposed model.

Keywords: *Many degrees of freedom (MDOF), model, mode, frequencies*

INTRODUCTION

Every structural system is essentially system with distributed mass i.e. system with infinite degrees of freedom. Dynamic analysis of system with infinite degrees of freedom is very tedious and involves voluminous numerical work [1-5]. In order to simplify its dynamic analysis, the system is idealized by transforming it to system with many degrees of freedom often abbreviated MDOF system [1-5, 6-8]. The latter is essentially the same structural system but having its entire mass substituted with lumped masses located at some chosen nodal points of the structure (Figure 1). The sum of these lumped masses necessarily is not equal to the entire mass of the system. The degree of freedom of the system following this simplification is numerically equal to the number of parameters essentially necessary to define the motion of the masses now considered as a system. MDOF model is said to be fully defined if the magnitudes of the lumped masses and their individual locations are precisely known [9]. The accuracy of the response of any simulating MDOF system depends mainly on adequate number of chosen nodal points, their locations as well as the magnitude of the nodal lumped masses [9].

The conventional approach [1-2,9] of determination of mass properties applies simple statics in evaluating nodal masses of individual beam segments which sum up to give the lumped masses (Figure 2). The flaws of the conventional approach consist in the fact that it doesn't take into consideration the physical properties of the structure coupled with the assumption that the lumped masses remain the same irrespective of the excitation frequency. These reduce the accuracy of the response of the MDOF model.

This paper therefore proposes a rational method of determination of mass properties of MDOF model simulating some system with distributed mass. The underlying hypothesis of this new method is that at any time t in the course of the system's motion, the kinetic energy as well as the nodal displacements of the actual system are the same as those of the simulating MDOF model. The accuracy of the present model will be verified by comparing its response to self excitation with those of the conventional model and the actual system, the latter serving as the control.

DYNAMICALLY EQUIVALENT MDOF MODEL

The actual system and its simulating MDOF model are said to be dynamically equivalent if their kinetic energies as well as their nodal displacements are respectively equal at any time t in the course of the system's motion.

Dr. N.N. Osadebe

Without loss of generality and for uniqueness of solution, the lumped masses of the present model are assumed to be of equal magnitude i.e.

$$m_1 = m_2 = \ldots = m_n = m \qquad (1)$$

Let the nodal displacement of the k^{th} node, the location of the lumped mass m_k, at time t be $y_k(t)$. Let $V_{(x,t)}$ also denote the displacement function of the actual system at time t. The above dynamic equivalence can be expressed mathematically as follows: -

$$\frac{1}{2}\int_0^L \rho(x)\dot{V}^2(x,t)d_x = \frac{1}{2}\sum_{k=1}^{n} m_k \dot{y}_k^2(t) \qquad (2a)$$

$$V(x,t) = \sum_{k=1}^{n} \psi_k(x)y_k(t) \qquad (2b)$$

where $\rho(x)$ is the mass per unit length of the system with distributed mass, and dot denotes differentiation with respect to the independent time variable t. In the above decomposition [Eqn. (2b)] each $\psi_k(x)$ is known, chosen in advance for any given beam or framed structure. It represents the elastic curve of the conjugate system of the investigated system due to a unit displacement of the k^{th} node.

$$\text{i.e. } y_k = 1,\ y_j = 0,\ j \neq k \qquad (3)$$

The conjugate system on the other hand is derived from the given system by introducing displacement restraints at each of the n nodal points as shown in Figure 3a and 3b.

After substituting Eqn (2b) into Eqn (2a) and integrating with respect to x the following expression is obtained.

$$\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}\,\dot{y}_j(t)\,\dot{y}_k(k) = \frac{1}{2}\sum_{k=1}^{n} m_k\,\dot{y}_k^2(t) \qquad (4)$$

where

$$a_{jk} = \int_{(L)} \rho(x)\,\psi_j(x)\,\psi_k(x)\,dx \qquad j = 1,2,3\ldots n \quad (5)$$

In matrix form Eqn (5) can be written as follows: -

$$\frac{1}{2}\dot{Y}^T A \dot{Y} = \frac{1}{2}\dot{Y}^T M \dot{Y} \qquad (6)$$

where

$$\dot{Y} = \begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \cdot \\ \dot{y}_n \end{Bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$M = \begin{bmatrix} m_1 & 0 & 0 & \ldots & 0 \\ 0 & m_2 & 0 & \ldots & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 0 & 0 & 0 & \ldots & m_n \end{bmatrix}$$

After differentiating both sides of Eqn (6) with respect to t the following equation is obtained.

$$A\ddot{Y} = M\ddot{Y} \qquad (7)$$

where

$$\ddot{Y} = \begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \cdot \\ \cdot \\ \ddot{y} \end{Bmatrix}$$

Eqn (7) implies that the forces of inertia of both the actual system and the simulating MDOF model at the nodal points are equal. Eqn (7) can be written as follows after noting Eqn (1).

$$B\ddot{Y} = \emptyset \qquad (8)$$

where

$$B = \begin{bmatrix} (a_{11}-m) & a_{12} & \ldots & a_{1n} \\ a_{21} & (a_{22}-m) & \ldots & a_{2n} \\ \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \ldots & \cdot \\ a_{n1} & \cdot & \ldots & (a_{nn}-m) \end{bmatrix}$$

For non-trivial solution of Eqn (8),

$$\det B = 0 \qquad (9)$$

Eqn (9) is an eigen-value problem leading to determination of various values of m

$$m^{(1)} > m^{(2)} > m^{(3)} \ldots > m^{(n)} \qquad (10)$$

Taking note of Eqn (1) each value of m generates a distinct mass vector

$$m = \begin{Bmatrix} m^{(i)} \\ m^{(i)} \\ \cdot \\ \cdot \\ \cdot \\ m^{(i)} \end{Bmatrix} \quad i = 1,2, \ldots n \qquad (11)$$

Each mass vector on the other hand satisfies the condition for dynamic equivalence for a particular mode of vibration as will be shown in the numerical example.

NATURAL FREQUENCIES

The frequency equation obtained through flexibility formulation is given by [1,9]

$$\det \begin{bmatrix} (\delta_{11}-k) & \delta_{12} & \cdots & \delta_{1n} \\ \delta_{21} & (\delta_{22}-k) & \cdots & \delta_{2n} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \delta_{n1} & \delta_{n2} & \cdots & (\delta_{nn}-k) \end{bmatrix} = 0 \qquad (12)$$

$$\text{where } k = \frac{1}{mw^2} \qquad (13)$$

and δ_{ij} is the displacement of the i^{th} nodal point due to a unit load applied at the j^{th} nodal point.

$$\delta_{ij} = \sum_{k=i}^{r} \int_{(L)} \frac{M_i M_j}{EI_k} dx \qquad (14)$$

In Eqn (14) above, M_i and M_j are bending moments due to unit load applied at k^{th} and j^{th} nodal points respectively. EI_k is the rigidity of the k^{th} beam element of the given structure.

Eqn (12) is an eigen-value problem leading to determination of n modal characteristic values.

$$K_1 > \quad K_2 > \; \ldots\ldots > K_n \qquad (15)$$

Using Eqn (13) and Eqn (10) the modal frequencies are evaluated as follows:

$$w_j = \sqrt{\frac{1}{K_j m^{(i)}}} \quad j = 1,2,3 \ldots n \qquad (16)$$

If the flexibility influence coefficients δ_{ij} used in Eqn (12) are multiples of the rigidity EI then the expression of w_j is

$$w_j = \sqrt{\frac{EI}{K_j m^{(i)}}} \quad j = 1,2,3 \ldots n \qquad (17)$$

where

$$w_1 < w_2 < w_3 \ldots < w_n$$

NUMERICAL EXAMPLE

A simply supported uniform beam shown in Figure 4 whose distributed mass intensity is = 4.75 kg/m is used for the numerical study. The associated MDOF model has three lumped masses m_1, m_2 and m_3. The degree of freedom of the model is three.

Using conventional method, the lumped masses are evaluated to be

$$m_1 = m_3 = 1.25\rho \, , \; m_2 = 1.5\rho \qquad (18)$$

Using the present method the determinant B (see Eq 8) for evaluating the modal lumped masses is obtained to be:

$$B = \begin{bmatrix} (1.17086-\frac{m}{\rho}) & 0.21355 & -0.12411 \\ 0.21355 & (1.26561-\frac{m}{\rho}) & 0.21355 \\ -0.12411 & 0.21355 & (1.7086-\frac{m}{\rho}) \end{bmatrix} \qquad (19)$$

Solution of Eqn. (9) for the above matrix gives the following values of m

$m^{(1)} = 1.4103025\rho \qquad m^{(2)} = 1.3137025\rho$

$$m^{(3)} = 0.883125\rho \tag{20}$$

Consequently, the lumped mass vectors necessary for calculation of the first, second and third modal frequencies are respectively

$$M = \begin{Bmatrix} 1.4103215\rho \\ 1.410325\rho \\ 1.410325\rho \end{Bmatrix} \tag{21}$$

$$M = \begin{Bmatrix} 1.3137025\rho \\ 1.3137025\rho \\ 1.3137025\rho \end{Bmatrix} \tag{22}$$

$$M = \begin{Bmatrix} 0.883125\rho \\ 0.883125\rho \\ 0.883125\rho \end{Bmatrix} \tag{23}$$

The frequency matrix (see Eqn (12) for the beam model is obtained to be

$$\begin{bmatrix} (1.066667-K) & 1.479167 & 0.766067 \\ 1.479167 & (2.604167-\beta K) & 1.479167 \\ 0.766667 & 1.479167 & (1.066667-K) \end{bmatrix} \tag{24}$$

where $$K \frac{EI}{Mw^2} \tag{25}$$

$\beta = \frac{1.25\rho}{1.5\rho} = 0.83333$ for the conventional MDOF model

and $\beta = 1$ for the present model.

Solution of Eqn (12) for the above matrix gives for the conventional model

$$K_1 = 4.85997 \quad K_2 = 0.299995 \quad K_3 = 0.09889 \tag{26}$$

For the present model the values of K are

$$K_1 = 4.34581 \quad K_2 = 0.299994 \quad K_3 = 0.091692 \tag{27}$$

The modal frequencies for the conventional model are evaluated using Eqn. (25).

$$W_j = \sqrt{\frac{EI}{1.25\rho K_j}} \qquad j = 1,2,3$$

For the present model, using Eqns (21), (22), (23) and (25) the modal frequencies are

$$W_j = \sqrt{\frac{EI}{m^{(i)}K_j}} \qquad j = 1,2,3$$

For the actual system i.e. system with distributed mass [12]

$$W_j = \frac{j^2\pi^2}{L^2}\sqrt{\frac{EI}{\rho}} \qquad j = 1,2,3.$$

All these results are shown in Table 1.

Table 1: Modal Frequencies

MODEL	NATURAL FREQUENCIES HZ		
	W_1	W_2	W_3
CONVENTIONAL	$0.4087\sqrt{\frac{EI}{\rho}}$	$1.6330\sqrt{\frac{EI}{\rho}}$	$2.8515\sqrt{\frac{EI}{\rho}}$
PRESENT	$0.4039\sqrt{\frac{EI}{\rho}}$	$1.5929\sqrt{\frac{EI}{\rho}}$	$3.5142\sqrt{\frac{EI}{\rho}}$
DISTRIBUTED MASS	$0.3948\sqrt{\frac{EI}{\rho}}$	$1.5791\sqrt{\frac{EI}{\rho}}$	$3.5531\sqrt{\frac{EI}{\rho}}$

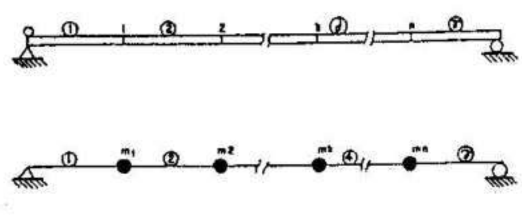

Figure 1: System with distributed mass and its related MDOF model

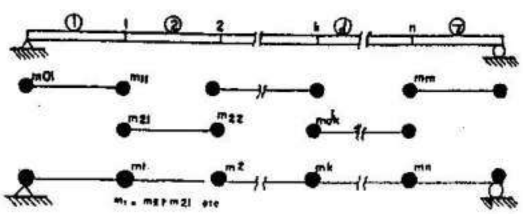

Figure 2: Lumping of mass at beam nodes by means of simple statics (Conventional method [2]

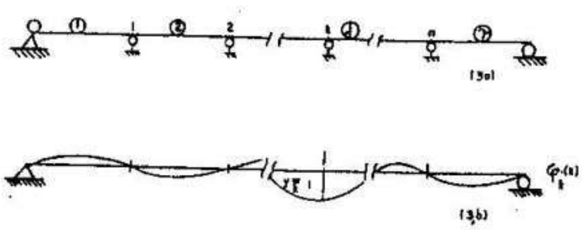

Figure 3: Conjugate system (3a) and its associated elastic curve $\varphi_k(x)$ (3b)

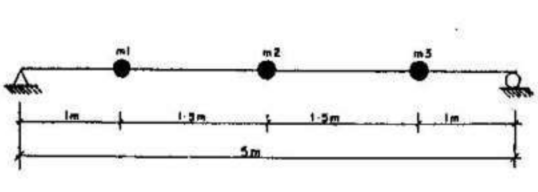

Figure 4: MDOF model of 3 degrees of freedom for numerical study

DISCUSSIONS OF RESULT AND CONCLUSION

From Table 1 it is evident that the present formulation which is based on the dynamic equivalence of simulating MDOF model and the actual system is more accurate than the conventional model. Its results are almost identical with those of the actual system showing high predictive capability of the present MDOF model. The difference in responses of both models is due to the difference in magnitude of their lumped masses. The lumped masses of the conventional model which are determined by simple statics are assumed constant irrespective of the excitation frequency. On the other hand the lumped masses of the present model are not constant but are dependent on excitation frequency. This shows that, for the simulating MDOF system to be dynamically equivalent with the actual system the lumped masses must be changing with the excitation frequency.

In conclusion, the determination of modal lumped masses demands the determination of the elastic curves, $\psi_j(x)$ of the conjugate system, [10, 11], setting up of the matrix B, and solution of the eigen-value problem arising from the annulment of the determinant of B [11]. All these mathematical processes have been fully computerized and can be accomplished in matter of minutes. A further work to be done is the unravelling of the implicit functional relationship existing between lumped masses and excitation frequencies. This will enable estimation of the lumped masses for forced vibration analysis of the system for excitation frequencies outside the range of the natural frequencies.

One good quality of the present formulation is that all the natural frequencies predictable by the model are almost identical with those of the actual system unlike the conventional model whose only first few frequencies are sufficiently accurate. This formulation can also be applied to framed structures.

NOTATION

MDOF -	many degrees of freedom
m_j -	lumped mass at the jth nodal point
ρ -	mass per unit length of the system with distributed mass
$V_{(x,t)}$ -	dynamic deflection function of the system with distributed mass
$y_k(t)$ -	time dependent nodal displacement of the kth nodal point
$\psi_k(t)$ -	elastic curve of the conjugate system due to a unit deflection of the k^{th} nodal point
$\{m\}_{nx1}$ -	mass vector
δ_{ij} -	displacement of ith node due to a unit load applied at j^{th} node
w, w_j -	frequency and nodal frequencies respectively.

REFERENCES

1. Varbanov, C.P. Stability and Dynamics of Elastic Systems, Technika Press, Sofia, 2nd ed, 1976.

2. Clough, R.W. and Penzien, J. Dynamics of Structures, McGraw Hill, Int. Students edition, Tomyo, 1982.

3. Newmark, N. and Hosenblueth, E. Fundamentals of Earthquake Engineering, Prentice-Hall, 1971.

4. Lin, V.K. Probabilistic Theory of Structural Dynamics, McGraw Hill, New York, 1967.

5. Howson, W.P. and Williams, F.W. Natural Frequencies of Frames with Axially Load. Timoshenko Members, Journal of Sound and Vibration, Vol. 26, pp. 503-515, 1973.

6. Varbanov, C.P. and Kapitanov, N. Effect of Axial Forces on the Dynamic Characteristics and Seismic Loads of Multi-Storey Frames. Annuaire De L'Institute D'Architecture et De Genie Civil, Sofia Vol. XXVII No. 3 pp. 7-14, 1980.

7. Osadebe, N.N. Vibrations of Multi-Storey Frames with Flexible Horizontal Members (Accepted for Publication), Transactions of Nigerian Society of Engineers, 1997.

8. Anya, C.U. Dynamic Analysis of Tall Buildings, M. Eng. Thesis, Dept. of Civil Engineering, University of Nigeria, Nsukka, Dec. 1995.

9. Bezukhov, N. et. al. Stability and Dynamics of Structures (learnt by example), Mir. Publishers, Moscow.

10. Kapitanov, N. and Kafelov, I. Spline. Functions in static analysis of one-dimensional system. Annuaire D'Linstitute D'Architecture et De Genie Civil, Sofia Vol. XXXI (5), pp. 83-91, 1984.

11. Karamanski, T.D. Numerical Methods in Structural Mechanics, Technika Press, Sofia, 1976.